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Real Analytic Eisenstein series of weight k and index m with respect to the Jacobi Group on $\mathrm{SL}_2(\mathbb{Z})$

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Abstract: The aim of this paper is to obtain the Fourier coefficients of the real analytic Eisenstein series of weight k and index m with respect to the Jacobi group of degree one on the full modular group. Moreover, we study the localization of a pole, its residue and Kronecker's limit formula of this series.

0. Introduction: Let k be an integer. For the sake of simplicity, let m be a square integer m_2^2 with an integer m_2. For each integer t with $t^2 \equiv 0 \mod 4m$ and $s \in \mathbb{C}$, let $\phi_{t,s}(\tau, s)$ be the function $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$ given by

$$(0,1) \qquad \phi_{t,s}(\tau, s) := e^m \left(\frac{t^2\tau}{4m^2} + \frac{tz}{m} \right) (\mathrm{Im}(\tau))^{s-\kappa}$$

where $\kappa = \frac{k-1/2}{2}$ and $e^m(\alpha) = \exp(2\pi i m\alpha)$. Let Γ be the full modular group $\mathrm{SL}_2(\mathbb{Z})$, Γ^J the Jacobi group $\{[M, (\lambda, \mu), \rho] | M \in \Gamma, \lambda, \mu, \rho \in \mathbb{Z}\}$ and $\Gamma^J_{\infty,+}$ the subgroup of Γ^J defined by

$$\Gamma^J_{\infty,+} = \{[(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}), (0, \mu), \varrho] | n, \mu, \rho \in \mathbb{Z}\}.$$

Following [EZ85] and [Ara90], we define the real analytic Eisenstein series $E_{k,m,t}((\tau, z), s)$ of weight k and index m with respect to the Jacobi group Γ^J of degree one by

$$(0,2) \qquad E_{k,m,t}((\tau, z), s) := \sum_{\gamma \in \Gamma^J_{\infty,+} \backslash \Gamma^J} (\phi_{t,s}|_{k,m}\gamma)(\tau, z) \quad ((\tau, z) \in \mathbb{H} \times \mathbb{C})$$

where $\gamma = (M, (\lambda, \mu), \varrho)$ and

$$(0,2,\mathrm{a}) \qquad t \in R^{\mathrm{null}} = \{t \in R | t^2 \equiv 0 \mod 4m\}, \quad R = \{t \in \mathbb{Z}/2m\mathbb{Z}\}.$$

From this definition the Eisenstein series $E_{k,m,t}((\tau, z), s)$ has the following expression

(0,3)

$$E_{k,m,t}((\tau, z), s) = \sum_{M \in \Gamma_{\infty,+} \backslash \Gamma} \sum_{l \in \mathbb{Z}} J(M, \tau)^{-k} (\mathrm{Im}(M\tau))^{s-\kappa}$$
$$\cdot e^m \left((l + \frac{t}{2m})^2 M\tau + 2(l + \frac{t}{2m}) \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right)$$

with $\Gamma_{\infty,+} = \{(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}), |n \in \mathbb{Z}\}$ and $M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$. We see easily from this expression that $E_{k,m,t}((\tau, z), s)$ is absolutely convergent for $\mathrm{Re}(s) > 5/4$.

Arakawa in [Ara90] studied the above real analytic Jacobi-Eisenstein series which are "natural generalization" of the holomorphic Eisenstein series of degree one. He obtained the analytic continuation and proved its functional equation. The key to his proof is to relate our real analytic Jacobi-Eisenstein series with those associated with theta multiplier systems after Roelcke[Roe], and then to make use of a general theory for real analytic Jacobi-Eisenstein series. He calculated explicitly only for square free m the Fourier coefficient of the real analytic Jacobi-Eisenstein series $E_{k,m,t}((\tau, z), s)$, in which case there is only one Eisenstein series.

B.Heim studied with him in [AH98] real analytic Jacobi-Eisenstein series of higher degree. W.Kohnen [Koh93] considered the lifting to the Siegel Eisenstein series with another real analytic Jacobi-Eisenstein series. T.Sugano [Sug95] gained also some results for a functional equation of Jacobi-Eisenstein series.

Our aim in this paper is to calculate directly Fourier coefficients for the Jacobi-Eisenstein series $E_{k,m,t}((\tau, z), s)$ with square m and to give its explicit formula, since the series is "natural generalization" of holomorphic one. Moreover, we want to study the localization of a pole, its residue and as an application Kronecker's limit formula.

The following properties are an immediate consequence of the definition (0,3):

i) $(E_{k,m,t}((\cdot,\cdot), s)|_{k,m}\gamma)(\tau, z) = E_{k,m,t}((\tau, z), s)$ for all $\gamma \in \Gamma^J$,

ii) If $k > 3$ and s is evaluated at $s = \kappa$, then $E_{k,m,t}((\tau, z), \kappa)$ coincides with the holomorphic Jacobi-Eisenstein series of Eichler-Zagier.

1. Definitions, Remarks and Main Theorems: We assume now that k is an integer (not necessarily positive). Let κ be the number given by

$$\frac{k - 1/2}{2}. \tag{1,0}$$

For the sake of simplicity, let m be a square m_2^2 with an integer m_2. R and R^{null} are as (0,2,a) and $e^m(\alpha)$ is an abbreviation of $\exp(2\pi i m\alpha)$ for any $\alpha \in \mathbb{C}$. Set for each $t \in R$,

$$\theta_t(\tau, z) := \sum_{l \in \mathbb{Z}} e^m \left(\tau(l + \frac{t}{2m})^2 + 2z(l + \frac{t}{2m}) \right) \quad (\tau \in \mathfrak{H}, z \in \mathbb{C}). \tag{1,1}$$

The followings are our main results:

Theorem. *For the real analytic Jacobi-Eisenstein series defined as above, the series $E^*_{k,m,t}((\tau,z),s) := \zeta(4s-1)E_{k,m,t}((\tau,z),s)$ has at least at $s=\frac{3}{4}$ a simple pole and its residue R^* is given by*

(1,2)

$$R^* = \frac{1}{4}R_0 \text{ with } R_0 = \frac{e^{-\pi ik/2}}{\sqrt{2m}}(\mathrm{Im}\ \tau)^{\frac{1}{4}-\kappa}\gamma(\frac{3}{4},\kappa)\sum_{\substack{r\in\mathbf{Z}\\ r^2\equiv 0 \mod 4m\\ D=0}}\theta_r(\tau,z)\cdot(\phi^{D=0}_{t,r}(\frac{3}{4})+(-1)^k\phi^{D=0}_{-t,r}(\frac{3}{4})).$$

Moreover, the following limit formula is valid:

$$\lim_{s\to\frac{3}{4}}\left\{E^*_{k,m,t}((\tau,z),s)-\frac{R^*}{s-3/4}\right\} = C_0+h_0 \tag{1,3}$$

where C_0 is a constant with respect to s and given by $C_0 = 2C_{\mathrm{Euler}}R_0$ with Euler's constant C_{Euler} and the number h_0 is given by

$$h_0 = \frac{\pi^2}{6}(\mathrm{Im}\ \tau)^{3/4-\kappa}\Theta_t(\tau,z)$$
$$+\frac{(\mathrm{Im}\ \tau)^{1-2\kappa}}{\sqrt{2im}}\sum_{\substack{D,r\in\mathbf{Z}\\ D<0\\ r^2\equiv 0 \mod 4m\\ D\equiv r^2 \mod 4m}} L_{D/4m}(1)V_{\frac{3}{4},\kappa}(\frac{D}{4m}\mathrm{Im}\tau)(\phi^{D<0}_{t,r}(\frac{3}{4})+(-1)^k\phi^{D<0}_{-t,r}(\frac{3}{4}))e(\frac{|D|}{4m}\mathrm{Re}\,\tau)\,\theta_r(\tau,z).$$

Here $\phi^{D\le 0}_{t,r}(s)$, $\gamma(s,\kappa)$, $L_{D/4m}$, $\Theta_t(\tau,z)$ and $V_{s,\kappa}$ are given in proposition 1.

For the proof we use proposition 1 and consider

$$C_0 = \frac{2e^{-\pi ik/2}}{\sqrt{2m}}\lim_{s\to\frac{3}{4}}\left\{\zeta(4s-2)(\mathrm{Im}\ \tau)^{1-s-\kappa}\gamma(s,\kappa)-\frac{(\mathrm{Im}\ \tau)^{\frac{1}{4}-\kappa}\gamma(\frac{3}{4},\kappa)/4}{s-3/4}\right\}$$
$$\cdot\sum_r\theta_r(\tau,z)\cdot(\phi^{D=0}_{t,r}(\frac{3}{4})+(-1)^k\phi^{D=0}_{-t,r}(\frac{3}{4})),$$

$$h(s) := \zeta(4s-1)(\mathrm{Im}\ \tau)^{s-\kappa}\Theta_t(\tau,z)$$
$$+\frac{(\mathrm{Im}\ \tau)^{1-2\kappa}}{\sqrt{2im}}\sum_{\substack{D,r\in\mathbf{Z}\\ D<0\\ r^2\equiv 0 \mod 4m\\ D\equiv r^2 \mod 4m}} L_{D/4m}(2s-\frac{1}{2})V_{s,\kappa}(\frac{D}{4m}\mathrm{Im}\tau)\theta_r(\tau,z)(\phi^{D<0}_{t,r}(s)+(-1)^k\phi^{D<0}_{-t,r}(s))e(\frac{|D|}{4m}\mathrm{Re}\,\tau),$$

$\zeta(2)=\frac{\pi^2}{6}$ and $h_0 := h(3/4)$.

Question: Is the function $\gamma(s,\kappa) = \frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)}$ with $s=\frac{3}{4}$ represented by log? Can h_0 be regarded as the generalization of the classical $\log|\eta(\tau)|$?

Proposition 1. *Let m be a square m_2^2 with a positive integer m_2 and k an integer. Let $t, r \mod 2m$ be integers with $t^2 \equiv r^2 \equiv 0 \mod 4m$. For the Jacobi-Eisenstein series $E_{k,m,t}((\tau,z),s)$ we have then the Fourier expansion*

$$E_{k,m,t}((\tau,z),s) = (\mathrm{Im}\ \tau)^{s-\kappa}\Theta_t(\tau,z) + (\mathrm{Im}\ \tau)^{1-(s+\kappa)} \sum_{\substack{r\in\mathbb{Z} \\ r^2\equiv 0 \mod 4m \\ D=0}} \Phi_{t,r}^{D=0}(s,\kappa)\theta_r(\tau,z)$$

$$+ (\mathrm{Im}\ \tau)^{1-2\kappa} \sum_{\substack{D,r\in\mathbb{Z} \\ D<0 \\ r^2\equiv 0 \mod 4m \\ D\equiv r^2 \mod 4m}} \Phi_{t,r}^{D<0}(s,\kappa)\ e(\frac{|D|}{4m}Re\tau)\,\theta_r(\tau,z)$$

where $\quad \Theta_t(\tau,z) := \theta_t(\tau,z) + (-1)^k\theta_{-t}(\tau,z), \quad \theta_t(\tau,z):$ as *(1,1)*,

$$\Phi_{t,r}^{D=0}(s,\kappa) = \frac{e^{-\pi ik/2}}{\sqrt{2m}}\gamma(s,\kappa)\Psi_{t,r}^{D=0}(s) \text{ with } \gamma(s,\kappa) = \frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)}$$

and $\quad \Phi_{t,r}^{D<0}(s,\kappa) = \frac{1}{\sqrt{2im}}V_{s,\kappa}\left(\frac{D}{4m}\mathrm{Im}\tau\right)\Psi_{t,r}^{D<0}(s).$

For $w \neq 0$, $V_{s,\kappa}(w)$ is defined by

$$V_{s,\kappa}(w) := \int_{\mathbb{R}} \frac{e\,(w\xi)\,d\xi}{(\xi+i)^{s+\kappa}(\xi-i)^{s-\kappa}}.$$

Here we have for the prime p with $p|m_2$

$$\Psi_{t,r}^{D=0}(s) = \frac{\zeta(4s-2)}{\zeta(4s-1)}(\phi_{t,r}^{D=0}(s) + (-1)^k\phi_{-t,r}^{D=0}(s))$$

with $\phi_{t,r}^{D=0}(s) := \{$*elementary expression of* $p^s\}$ *(see Prop. in* **2**)

and $\Psi_{t,r}^{D<0}(s) = \dfrac{L_{D/4m}(2s-1/2)}{\zeta(4s-1)}(\phi_{t,r}^{D<0}(s) + (-1)^k\phi_{-t,r}^{D<0}(s))$

with $\phi_{t,r}^{D<0}(s) := \{$*elementary expression of* $L_{\Delta_p,p}(2s-1/2)$ *and* $p^s\}$ *(see Prop. in* **2**)

where $\Delta_p = \frac{D}{4mp^{2n'}}$ with $n' = \mathrm{ord}_p(t_1 - r_1)$, $t_1 = t/2m_2$, $r_1 = r/2m_2$ and $L_{\Delta_p,p}$ is the p-part of L-function L_{Δ_p}.

Remark: 1) If $\Delta \equiv 0, 1 \mod 4$ and $\Delta \neq 0$, we write $\Delta = D_0 f^2 (f \in \mathbb{N})$. For $\Delta \in \mathbb{Z}$ we have

$$L(s,\Delta) = \begin{cases} 0 \text{ if } \Delta \equiv 2 \text{ or } 3 \mod 4, \\ \zeta(2s-1) \text{ if } \Delta = 0, \\ L_{D_0}(s)\gamma_{D_0}^s(f) \text{ if } \Delta \equiv 0, 1 \mod 4 \end{cases}$$

where $\gamma^s_{D_0}(f) = \sum_{d|f} \mu(d)\varepsilon_{D_0}(d)d^{-s}\sigma_{1-2s}(f/d)$ *with* $\sigma_s(d) = \sum_{d'|d} d'^s$. D_0 *is the discriminant of* $\mathbb{Q}(\sqrt{\Delta})$ *and* $\epsilon_{D_0}(\cdot)$ *the Kronecker symbol with discriminant* D_0. *The function* $L^*_D(s) := L_D(s)(2\pi)^{-s}|D|^{s/2}\Gamma(s)$ $(D < 0)$ *satisfies the funtional equation* $L^*_D(s) = L^*_D(1-s)$.

2) If $V^*_{s,\kappa}(w)$ is defined by

$$V^*_{s,\kappa}(w) := \frac{-\Gamma(s-\kappa)}{(\pi|w|)^{s-\kappa}} V_{s,\kappa}(w),$$

then we can rewrite $V^*_{s,\kappa}(w)$ in the form

$$V^*_{s,\kappa}(w) = \int_0^\infty u^{2s-1} e^{-\pi w(u+1/u)} \int_{-\infty}^\infty e^{-\pi w v^2} \left(v + \frac{u^{1/2}+u^{-1/2}}{i}\right)^{-2\kappa} dv \frac{du}{u}.$$

$V^*_{s,\kappa}(w)$ is an entire function and satisfies a functional equation: $V^*_{s,\kappa}(w) = \mathrm{sgn}(w) V^*_{1-s,\kappa}(w)$. $V^*_{s,0}(w) = K_{1/2}$ is the K-Bessel function (s. [GZ86]).

Examples for Theorem 1: We see examples of terms $(\Psi^{D=0}_{t,r})_{t,r}$ and $\Phi^{D<0}_{t,r}$:

$$\begin{aligned}
(\Phi^{D=0}_{t,r})_{t,r} &= \frac{e^{-\pi i k/2}\gamma(s,\kappa)}{\sqrt{2m}} (\Psi^{D=0}_{t,r})_{t,r} \text{ and} \\
\Phi^{D<0}_{t,r}(s,\kappa) &= \frac{1}{\sqrt{2im}} V_{s,\kappa}\left(\frac{D}{4m}\mathrm{Im}\tau\right) \Psi^{D<0}_{t,r}(s) \\
&\text{with } \Psi^{D\le 0}_{t,r} = \psi^{D\le 0}_{t,r} + (-1)^k \psi^{D\le 0}_{-t,r}
\end{aligned}$$

where $t^2 \equiv r^2 \equiv 0 \mod 4m$, $t = 2m_2t_1$ and $r = 2m_2r_1$. Let be $\zeta_p(s) := 1 - p^{-s}$ and $L_{\Delta_p,p}(s)$ the p-part of $L_{\Delta_p}(s)$. For $\Psi^{D\le 0}_{t,r}$ we confer also Lemma and Proposition 2 in **2**.

I) For $m = 2^2$ i.e. $m_2 = 2$ and $t_1, r_1 = 0, 1$, then $t, r = 0, 4$ $(t = 4t_1, r = 4r_1)$, the constant term matrix (the case $D = 0$) is

$$\begin{pmatrix} \Psi_{0,0} & \Psi_{0,4} \\ \Psi_{4,0} & \Psi_{4,4} \end{pmatrix} \text{ with } \Psi_{r,t} := \Psi^{D=0}_{r,t}$$

where $\quad \Psi^{D=0}_{0,4} = \Psi^{D=0}_{4,0} = (1+(-1)^k)\dfrac{\zeta(4s-2)}{\zeta(4s-1)}$ and

$$\Psi^{D=0}_{0,0} = \Psi^{D=0}_{4,4} = (1+(-1)^k)\frac{\zeta(4s-2)}{\zeta(4s-1)}\left\{\frac{\zeta_2(4s-1)\cdot\zeta_2(2s-3/2)}{\zeta_2(4s-2)\cdot\zeta_2(2s-1/2)\cdot\zeta_2(4s-3)} + 2^{3-4s}\right\}.$$

Next we see non-constant terms. Since in the case $D=-16$ the number $D/4m=-1$ is $\equiv 3 \mod 4$ and $(D/4m,m)=1$. Then we have

$$\Psi_{0,4}^{D=-16}=\Psi_{4,0}^{D=-16}=\Psi_{0,4}^{D=-16}=\Psi_{4,0}^{D=-16}=(1+(-1)^k)\frac{1}{\zeta(2s-1/2)}.$$

Case $D=-48$: In this case $D/4m$ is -3 and $(D/4m,m)=1$. So we have

$$\Psi_{0,0}^{D=-48}=\Psi_{4,4}^{D=-48}=\Psi_{4,0}^{D=-48}=\Psi_{0,4}^{D=-48}=(1+(-1)^k)\frac{L_{-3}(2s-1/2)}{\zeta(4s-1)}.$$

II) For $m=3^2$ i.e. $m_2=3$ we have $t_1,r_1=0,1,2$ and $t,r=0,6,12$ $(t=6t_1, r=6r_1)$ the constant term matrix is

$$\begin{pmatrix}\Psi_{0,0} & \Psi_{0,6} & \Psi_{0,12}\\ \Psi_{6,0} & \Psi_{6,6} & \Psi_{6,12}\\ \Psi_{12,0} & \Psi_{12,6} & \Psi_{12,12}\end{pmatrix} \text{ with } \Psi_{r,t}:=\Psi_{r,t}^{D=0}$$

where $\Psi_{0,6}^{D=0}=\Psi_{0,12}^{D=0}=\Psi_{6,0}^{D=0}=\Psi_{12,12}^{D=0}=(1+(-1)^k)\dfrac{\zeta(4s-2)}{\zeta(4s-1)}$,

$$\Psi_{6,6}^{D=0}=\Psi_{12,12}^{D=0}=\frac{\zeta(4s-2)}{\zeta(4s-1)}\{1+(-1)^k\phi_{0,0}^{D=0}\},$$

$$\Psi_{0,0}^{D=0}=(1+(-1)^k)\frac{\zeta(4s-2)}{\zeta(4s-1)}\cdot\phi_{0,0}^{D=0},\quad \Psi_{6,12}^{D=0}=\Psi_{12,6}^{D=0}=\frac{\zeta(4s-2)}{\zeta(4s-1)}\{\phi_{0,0}^{D=0}+(-1)^k\}$$

with $\phi_{0,0}^{D=0}=\dfrac{\zeta_3(2s-3/2)\cdot\zeta_3(4s-1)}{\zeta_3(4s-2)\cdot\zeta_3(4s-3)\zeta_3(2s-1/2)}+3^{3-4s}$.

Case $D=-16\cdot 3^2$: Since $D/4m=-4$ and $(D/4m,m)=1$ we have

$$\Psi_{t,r}^{D=-16\cdot 3^2}=\psi_{t,r}^{D=-16\cdot 3^2}+(-1)^k\psi_{-t,r}^{D=-16\cdot 3^2}=(1+(-1)^k)\frac{L_{-4}(2s-1/2)}{\zeta(4s-1)}\quad\text{for all } t,r.$$

Case $D=-12\cdot 3^4: D/4m=-3\cdot 3^2, \Delta=\dfrac{D}{4m}/3^2=-3$

where $\Psi_{0,6}^{D=-12\cdot 3^4}=\Psi_{12,0}^{D=-12\cdot 3^4}=\Psi_{6,0}^{D=-12\cdot 3^4}=\Psi_{0,12}^{D=-12\cdot 3^4}=(1+(-1)^k)\dfrac{L_{-3\cdot 3^2}(2s-1/2)}{\zeta(4s-1)}$,

$$\Psi_{0,0}^{D=-12\cdot 3^4}=(1+(-1)^k)\frac{L_{-3\cdot 3^2}(2s-1/2)}{\zeta(4s-1)}\phi_{0,0}^{-12\cdot 3^4},$$

$$\Psi_{6,6}^{D=-12\cdot 3^4}=\Psi_{12,12}^{D=-12\cdot 3^4}=(1+(-1)^k\phi_{0,0}^{-12\cdot 3^4})\frac{L_{-3\cdot 3^2}(2s-1/2)}{\zeta(4s-1)},$$

$$\Psi_{6,12}^{D=-12\cdot 3^4}=\Psi_{12,6}^{D=-12\cdot 3^4}=(\phi_{0,0}^{-12\cdot 3^4}+(-1)^k)\frac{L_{-3\cdot 3^2}(2s-1/2)}{\zeta(4s-1)}$$

with $\phi_{0,0}^{-12\cdot 3^4}:=\dfrac{L_{-3,3}(2s-1/2)}{L_{-3\cdot 3^2,3}(2s-1/2)}\dfrac{\zeta_3(2s-3/2)\cdot\zeta_3(4s-1)}{\zeta_3(4s-3)\zeta_3(2s-1/2)}+3^{3-4s}$.

In the case $D = -12 \cdot 3^2$ we have $D/4m = -3$, but $3^2 \nmid \frac{D}{4m}$. Then we have

$$\Psi_{0,0}^{D=-12\cdot 3^2} = (1+(-1)^k)\frac{1}{\zeta_3(2s-1/2)}\frac{\zeta_3(2s-3/2)}{\zeta_3(4s-3)}$$

and for another r, t analogous.

III) Case $m = 6^2$: In this case we have $m_2 = 6$, $t_1, r_1 \mod 6$ and then $t, r = 0, 12, 24, 36, 48, 60$ ($t_1 \mp r_1 = 0, 1, \dots, 5$). Let be

$$K_p := \frac{\zeta_p(2s-3/2)\cdot\zeta_p(4s-1)}{\zeta_p(4s-2)\cdot\zeta_p(4s-3)\cdot\zeta_p(2s-1/2)} + p^{3-4s} \text{ for } p = 2, 3.$$

Then we have

$$\phi_0 = K_2 \cdot K_3, \quad \phi_1 = \phi_5 = 1, \quad \phi_2 = \phi_4 = K_2, \quad \phi_3 = K_3.$$

$$\text{and } \Psi_{i,j} := \Psi_{12t_1,12r_1} = \frac{\zeta(4s-2)}{\zeta(4s-1)}\phi_{ij}, \; \phi_{ij} := \phi_i + (-1)^k\phi_j$$

$$\Psi_i := \Psi_{12t_1,12t_1} = \frac{\zeta(4s-2)}{\zeta(4s-1)}\phi_{ii}(s), (i,j = 0,1,2,3, \quad i := t_1 - r_1, j = -t_1 - r_1),$$

Therefore, the constant term matrix $(\Psi_{t,r}^{D=0})$ is given by

$$\begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 & \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_1 & \Psi_{02} & \Psi_{13} & \Psi_2 & \Psi_{31} & \Psi_{20} \\ \Psi_2 & \Psi_{13} & \Psi_{02} & \Psi_1 & \Psi_{20} & \Psi_{31} \\ \Psi_3 & \Psi_2 & \Psi_1 & \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_2 & \Psi_{31} & \Psi_{20} & \Psi_1 & \Psi_{03} & \Psi_{13} \\ \Psi_1 & \Psi_{20} & \Psi_{31} & \Psi_2 & \Psi_{13} & \Psi_{02} \end{pmatrix}.$$

For the case $D/4m = -4 \cdot 45$, we have $\frac{D}{4m}/3^2 = -20$. For this case we see two examples:

$$\Psi_{180,0}^{-4\cdot 3\cdot 180^2}(s) = (1+(-1)^k)\frac{L_{-180}(2s-1/2)}{\zeta(4s-1)},$$

$$\Psi_{0,0}^{-4\cdot 3\cdot 180^2}(s) = (1+(-1)^k)\frac{L_{-180}(2s-1/2)}{\zeta(4s-1)}\phi_{0,0}^{-4\cdot 3\cdot 180^2}(s)$$

where

$$\phi_{0,0}^{-4\cdot 3\cdot 180^2}(s) = \left\{\frac{L_{-20,2}(2s-1/2)}{L_{-180,2}(2s-1/2)}\cdot\frac{\zeta_2(2s-3/2)\cdot\zeta_2(4s-1)}{\zeta_2(4s-3)\zeta_2(2s-1/2)}\right\}$$

$$\cdot\left\{\frac{L_{-20,3}(2s-1/2)}{L_{-180,3}(2s-1/2)}\frac{\zeta_3(2s-3/2)\cdot\zeta_3(4s-1)}{\zeta_3(4s-3)\zeta_3(2s-1/2)} + 3^{2(3/2-2s)}\right\}.$$

IV) Case $m = 5^4 \cdot 7^4$: In this case we have $m_2 = 5^2 \cdot 7^2$, $t_1, r_1 \mod 5^2 \cdot 7^2$ and then $t, r = 0, 30\cdot 7^2, 2\cdot 30\cdot 7^2, \dots, (7^2-1)\cdot 30\cdot 7^2$, $(t_1 \mp r_1 = 0, 1, \dots, 7^2-1)$. Here we see the case $D/4m = 15r_1^2 - n = -3\cdot 5^4\cdot 7^4$. In this case we set $\Delta_p = \frac{D}{4m}/p^4 = -3\cdot 5^4\cdot 7^4/p^4$ and

$$\Psi_{0,0}^{-3\cdot 5^4\cdot 7^4\cdot 4\cdot 5^4\cdot 7^4}(s) = (1+(-1)^k)\frac{L_{-3\cdot 5^4\cdot 7^4}(2s-1/2)}{\zeta(4s-1)} \cdot K_5 \cdot K_7$$

$$\text{with } K_p := \frac{L_{-3\cdot 5^4\cdot 7^4/p^4,p}(2s-1/2)}{L_{-3\cdot 5^4\cdot 7^4,p}(2s-1/2)} \frac{\zeta_p(2s-3/2)}{\zeta_p(4(2s-3/2))} \frac{\zeta_p(4s-1)}{\zeta_p(2s-1/2)} + p^{6-8s}.$$

2. Proof of the Proposition 1:

For the begin of our calculation we follow the method of [EZ85] and [Ara90]. Now we see the Fourier expansion of

$$(2,1)\quad E_{k,m,t}((\tau,z),s) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn=r^2}} c_{n,r}(\eta;s)e(n\xi+rz) + \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>r^2}} c_{n,r}(\eta;s)e(n\xi+rz)$$

where $\tau = \xi + i\eta$. The constant terms is the partial sum

$$(2,2)\qquad \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn=r^2}} c_{n,r}(\eta;s)e(n\xi+qz).$$

As usual we devide the sum on the right side of the identity (2,2) in two parts according as $c = 0$ of $a \neq 0$, and using the identity

$$X^2M\tau + 2X\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{(z-X/c)^2}{\tau+d/c} + \frac{aX^2}{c}$$

we have

$$(2,3)\qquad E_{k,m,t}((\tau,z),s) = E_t^{c=0}((\tau,z),s) + E_t^{c\neq 0}((\tau,z),s)$$

with

(2,4)

$$E_t^{c=0}((\tau,z),s) = \eta^{s-\kappa}\Theta_t(\tau,z),$$

$$E_t^{c\neq 0}((\tau,z),s) = \sum_{\substack{(c,d)=1\\ c\neq 0}}\sum_{q\in\mathbb{Z}} \frac{\eta^{s-\kappa}}{(c\tau+d)^k|c\tau+d|^{2(s-\kappa)}} \cdot e^m\left(\frac{(z-(q+t/2m)/c)^2}{\tau+d/c} + \frac{a}{c}(q+\frac{t}{2m})^2\right)$$

where an integer a is chosen so that $ad \equiv 1 \mod c$ for coprime integers $c, d (c \neq 0)$. Now we define $F((\tau, z), s)$ by

$$F((\tau,z),s) = \sum_{p,q\in\mathbb{Z}} \frac{1}{(\tau+p)^k|\tau+p|^{2(s-\kappa)}} e^m\left(-\frac{(z+q)^2}{\tau+p}\right),$$

which is absolutely convergent for $\mathrm{Re}(s) > 3/4$. Replacing q by $\lambda - cq'$ ($q' \in \mathbb{Z}, \lambda \mod c$) on the right hand side of (2,4), we get

(2,5)

$$\begin{aligned} E_t^{c\neq 0}((\tau,z),s) = &\sum_{c=1}^{\infty} \sum_{\substack{d(c)\\(d,c)=1}} \sum_{\lambda(c)} \frac{\eta^{s-\kappa}}{c^{2s+1/2}} \\ &\cdot\left\{ e^m\left(\frac{a}{c}(\lambda+\frac{t}{2m})^2\right) F((\tau+\frac{d}{c}, z-\frac{1}{c}(\lambda+\frac{t}{2m})), s)\right. \\ &\left.+(-1)^k e^m\left(\frac{a}{c}(\lambda-\frac{t}{2m})^2\right) F((\tau+\frac{d}{c}, z-\frac{1}{c}(\lambda-\frac{t}{2m})), s)\right\} \end{aligned}$$

Since $F((\tau, z), s)$ is periodic in τ and z with period 1 and therefore has the Fourier expansion of the form

$$F((\tau,z),s) = \sum_{n,r\in\mathbb{Z}} \gamma_{n,r}(\eta,s)e(n\xi+rz) \quad (\tau=\xi+i\eta\in\mathbb{H}, z\in\mathbb{C}) \tag{2,6}$$

$$\text{with } \gamma_{n,r} = \int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{-k}|\tau|^{-2(s-\kappa)}e(-mz^2/\tau - n\xi - rz)dxd\xi \quad (z=x+iy).$$

Integrating with respect to x and changing the variable with $\xi \to \eta\xi$, we gain

$$\begin{aligned} \gamma_{n,r}(\eta,x) &= \int_{\mathbb{R}} (\tau/2im)^{1/2}\tau^{-k}|\tau|^{-2(s-\kappa)}e(\frac{r^2\tau}{4m} - n\xi)d\xi \\ &= \frac{\eta^{1-2s}}{\sqrt{2im}}\exp\left(-\frac{\pi}{2m}\eta r^2\right)\cdot V_{s,\kappa}\left((\frac{r^2}{4m}-n)\eta\right) \end{aligned}$$

where

$$V_{s,\kappa}(w) = \int_{\mathbb{R}} \frac{e(w\xi)d\xi}{(\xi+i)^{s+\kappa}(\xi-i)^{s-\kappa}} = \int_{\mathbb{R}} \frac{-e(-w\xi)d\xi}{(\xi+i)^{-2\kappa}(\xi^2+1)^{s+\kappa}} \quad (w\neq 0) \tag{2,7a}$$

and

$$V_{s,\kappa}(0) = e^{-\pi i\kappa}\gamma(s,\kappa) \text{ with } \gamma(s,\kappa) = \frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)} \quad \text{if } 4mn = r^2. \tag{2,7b}$$

So the Fourier coefficient $\gamma_{n,r}(\eta,s)$ is given as follows:

$$(2,8)\qquad \gamma_{n,r}(\eta,s)=\begin{cases}\dfrac{\eta^{1-2s}}{\sqrt{2m}}e^{-\pi ik/2}\gamma(s,\kappa)\exp\left(-\dfrac{\pi}{2m}\eta r^2\right) & \text{if } 4mn=r^2,\\ \dfrac{\eta^{1-2s}}{\sqrt{2im}}\exp\left(-\dfrac{r^2\pi}{2m}\eta\right)\cdot V_{s,\kappa}\left(\left(\dfrac{r^2}{4m}-n\right)\eta\right) & \text{if } 4mn>r^2\end{cases}$$

(cf: [Ara90] p.144, [EZ85] p.19, [GZ86] p.277-280). Since

(2,9)

$$E_t^{c\neq 0}((\tau,z),s)=\sum_{c=1}^{\infty}\sum_{\substack{d\bmod c\\(d,c)=1}}\sum_{\lambda\bmod c}\frac{\eta^{s-\kappa}}{c^{2s+1/2}}$$
$$\cdot\left\{e^m\left(\frac{a}{c}(\lambda+\frac{t}{2m})^2\right)F((\tau+\frac{d}{c},z-\frac{1}{c}(\lambda+\frac{t}{2m})),s)\right.$$
$$\left.+(-1)^k e^m\left(\frac{a}{c}(\lambda-\frac{t}{2m})^2\right)F((\tau+\frac{d}{c},z-\frac{1}{c}(\lambda-\frac{t}{2m})),s)\right\}$$

where

(2,10)

$$F((\tau+\frac{d}{c},z-\frac{1}{c}(\lambda+\frac{t}{2m})),s)=\sum_{n,r\in\mathbf{Z}}\gamma_{n,r}(\eta,s)e\left(n(\xi+\frac{d}{c})+r(z-\frac{1}{c}(\lambda+\frac{t}{2m}))\right)$$

$$\text{with } \gamma_{n,r}(\eta,s)=\begin{cases}\dfrac{\eta^{1-2s}}{\sqrt{2m}}e^{-\pi ik/2}\gamma(s,\kappa)\exp\left(-\dfrac{\pi}{2m}\eta r^2\right) & \text{if } 4mn=r^2,\\ \dfrac{\eta^{1-2s}}{\sqrt{2im}}V_{s,\kappa}\left(\left(\dfrac{r^2}{4m}-n\right)\eta\right)\exp\left(-\dfrac{\pi}{2m}\eta r^2\right) & \text{if } 4mn>r^2.\end{cases}$$

According to (2,4) (2,5), (2,6), (2,7) and (2,8) the Fourier expansion of $E_t^{c\neq 0}((\tau,z),s)$ is given explicitly by

(2,11)

$$\frac{\eta^{1-2s}}{\sqrt{2im}}\sum_{\substack{n,r\in\mathbf{Z}\\4nm-r^2\geq 0}}V_{s,\kappa}\left(\left(\frac{r^2}{4m}-n\right)\eta\right)(\psi_{t,r}(s)+(-1)^k\psi_{-t,r}(s))\cdot e\left(\frac{4mn-r^2}{2m}\xi\right)\theta_r(\tau,z).$$

Accordingly by (2,4) (2,8) and (2,11), the constant terms of $E_{k,m,t}$ equal

$$(2,12b)\quad \eta^{s-\kappa}\Theta_t(\tau,z)+\eta^{1-s-\kappa}\frac{\gamma(s,\kappa)}{e^{\pi ik/2}\sqrt{2m}}\sum_{\substack{n,r\in\mathbf{Z}\\4nm-r^2=0}}(\psi_{t,r}(s)+(-1)^k\psi_{-t,r}(s))\cdot\theta_r(\tau,z).$$

Here $\psi_{t,r}(s)$ $(t, r \mod 2m,\ t^2 \equiv r^2 \equiv 0 \mod 4m)$ is the Dirichlet series defined by (2,12a)

$$\psi_{t,r}(s) = \sum_{c\geq 1} \frac{1}{c^{2s+1/2}} \sum_{\substack{d(c)\\(d,c)=1}} G(t,r;c,d)$$

$$\text{with } G(t,r;c,d) := \sum_{\lambda \mod c} e_c\left(am(\lambda + \frac{t}{2m})^2 - r(\lambda + \frac{t}{2m}) + dn\right),$$

with an integer a given by $ad \equiv 1 \mod c$. Substituting $\lambda + \frac{t}{2m} \to d(\lambda + \frac{t}{2m})$ we have

$$G(t,r;c,d) = \sum_{\lambda \mod c} e_c\left(dm(\lambda + \frac{t}{2m})^2 - dr(\lambda + \frac{t}{2m}) + dn\right)$$

where we used $ad \equiv 1 \mod c$, $t^2 \equiv 0 \mod 4m$, $rt \equiv 0 \mod 2m$, $\frac{t}{2} \in \mathbb{Z}$ and $m(\lambda + \frac{t}{2m})^2, r(\lambda + \frac{t}{2m}) \in \mathbb{Z}$. So we gain

$$G(t,r;c,d) = \sum_{\lambda \mod c} e_{cm}\left(d(m\lambda + \frac{t}{2})^2 - dr(m\lambda + \frac{t}{2}) + dmn\right)$$

and then

$$\text{(2,13)} \quad \sum_{\substack{d \mod c\\(d,c)=1}} G(t,r;c,d) = \sum_{\substack{\lambda,d \mod c\\(d,c)=1}} e_{cm}(dQ(m\lambda + \frac{t}{2})) \text{ with } Q(\lambda) = \lambda^2 - r\lambda + mn.$$

Setting $D := r^2 - 4mn$ and putting together above formulas, we obtain the Fourier expansion of $E_{k,m,t}((\tau,z),s)$:

(2,14)

$$E_{k,m,t}((\tau,z),s) = (\operatorname{Im}\tau)^{s-\kappa}\Theta_t(\tau,z) + (\operatorname{Im}\tau)^{1-(s+\kappa)} \sum_{\substack{D,r\in\mathbb{Z}\\D=0}} \Phi_{t,r}^{D=0}(s)\theta_r(\tau,z)$$

$$+ (\operatorname{Im}\tau)^{1-(s+\kappa)} \sum_{\substack{D,r\in\mathbb{Z}\\D<0\\D\equiv r^2 \mod 4m\\r^2\equiv 0 \mod 4m}} \Phi_{t,r}^{D<0}(s)e\left(\frac{|D|}{4m}\mathrm{Re}\tau\right)\theta_r(\tau,z)$$

where $V_{s,\kappa}(w) = \displaystyle\int_{\mathbb{R}} \frac{e(w\xi)\,d\xi}{(\xi+i)^{s+\kappa}(\xi-i)^{s-\kappa}} (w \neq 0)$,

$$\Phi_{t,r}^{D=0}(s) = \frac{\gamma(s,\kappa)}{e^{\pi i k/2}\sqrt{2m}}(\psi_{t,r}^{D=0}(s) + (-1)^k\psi_{-t,r}^{D=0}(s)),$$

$$\Phi_{t,r}^{D<0}(s) = \frac{V_{s,\kappa}(\frac{D}{4m}\mathrm{Im}\tau)}{\sqrt{2im}}(\psi_{t,r}^{D<0}(s) + (-1)^k\psi_{-t,r}^{D<0}(s))$$

and $\psi_{t,r}^{D\leq 0}(s) := \displaystyle\sum_{c\geq 1} \frac{1}{c^{2s+1/2}} \sum_{\substack{\lambda,d \mod c\\(d,c)=1}} e_{cm}(dQ(m\lambda + \frac{t}{2}))$ with $Q(\lambda) = \lambda^2 - r\lambda + mn$.

Now we calculate $\psi_{t,r}^{D\leq 0}$ and $\psi_{-t,r}^{D\leq 0}$. Since we have for $Q(\lambda)=\lambda^2-r\lambda+mn$

(2,15a)

$$\psi_{t,r}^{D\leq 0}(s)=\sum_{c\geq 1}\frac{1}{c^{2s+1/2}}\sum_{\substack{\lambda,d \bmod c\\(d,c)=1}} e_{cm}(dQ(m\lambda+\frac{t}{2}))=\sum_{c\geq 1}\frac{1}{c^{2s+1/2}}\sum_{b|c}\mu(\frac{c}{b})b\sum_{\substack{\lambda'(b)\\ \lambda=m\lambda'\pm\frac{t}{2}\\ Q(\lambda)\equiv 0(bm)}}1$$

$$=\sum_{c\geq 1}\frac{1}{c^{2s-1/2}}\sum_{b|c}\mu(\frac{c}{b})N_{bm,t}(Q)=\zeta(2s-1/2)^{-1}\sum_{b\geq 1}\frac{N_{bm,t}(Q)}{b^{2s-1/2}}$$

where we used $\sum_{c'}\mu(c')c'^{-2s+1/2}=\zeta(2s-1/2)^{-1}$ with $c=bc'$. Here we set $N_{bm,t}(Q):=\sharp\{\lambda(b)|Q(m\lambda+\frac{t}{2})\equiv 0 \mod bm\}$. Since from $m=m_2^2$ and $t^2\equiv r^2\equiv 0 \mod 4m$ we have $t=2m_2t_1$, $r=2m_2r_1$ with $t_1,r_1 \mod m_2$:

(2,15b)

$$N_{bm,t}(Q):=\sharp\{\lambda(b)|Q(m\lambda+\frac{t}{2})\equiv 0 \mod bm\}=\sharp\{\lambda(b)|(m\lambda+\frac{t-r}{2})^2\equiv\frac{D}{4} \mod bm\}$$

$$=\sharp\{\lambda(b)|(m_2\lambda+(t_1-r_1))^2\equiv\frac{D}{4m} \mod b\}$$

$$=: N_{b,m,t_1-r_1}(D/4m) \text{ where } \frac{D}{4}\in\mathbb{Z} \text{ because } \frac{r}{2}\in\mathbb{Z}.$$

Putting $Z_{m,t_1-r_1}^{D\leq 0}(s):=\sum_{b\geq 1}\frac{N_{b,m,t_1-r_1}(D/4m)}{b^s}$ and $Z_{m,t_1-r_1,p}^{D\leq 0}(s):=\sum_{n\geq 0}\frac{N_{p^n,m,t_1-r_1}(D/4m)}{p^{ns}}$ we have

(2,16)

$$\psi_{t,r}^{D<0}(2s-1/2)=\zeta(2s-1/2)^{-1}Z_{m,t_1-r_1}^{D<0}(2s-1/2)$$
$$\psi_{t,r}^{D=0}(2s-1/2)=\zeta(2s-1/2)^{-1}Z_{m,t_1-r_1}^{D=0}(2s-1/2)$$
$$\text{with } Z_{m,t_1-r_1}^{D\leq 0}(2s-1/2)=\prod_p Z_{m,t_1-r_1,p}^{D\leq 0}(2s-1/2).$$

Setting $n_2:=\mathrm{ord}_p m_2$ and $n':=\mathrm{ord}_p(t_1-r_1)$ (i.e. $n'\leq n_2$) for $p|m_2$ we have a

Lemma. *For $n\geq 0$ let be $N_{p^n,m,A}(D):=\sharp\{\lambda \mod p^n|(m_2\lambda+A)^2\equiv D \mod p^n\}$ and $N_{p^n}(D):=\sharp\{\lambda \mod p^n|\lambda^2\equiv D \mod p^n\}$. Then we have for $n':=\mathrm{ord}_pA$ with $n'\leq n_2$ following values:*

A) Case $D=0$:

$$N_{p^n,m,A}(0)=\begin{cases} p^{2n_2}N_{p^n}(0) \text{ if } n\geq 2n_2=2n',\\ p^n \text{ if } n\leq 2n'\leq 2n_2,\\ 0 \text{ otherwise.}\end{cases}$$

B) Case $D < 0$:

$$N_{p^n,m,A}(D) = \begin{cases} N_{p^n}(D) \text{ if } (m_2,p)=1, \\ p^{2n_2} N_{p^{n-2n_2}}(D/p^{2n_2}) \text{ for } n > 2n_2 = 2n' \text{ if } n_2 = n' \text{ with } p^{2n_2}|D, \\ p^{2n'} \phi_{p,n_2-n'} N_{p^{n-2n'}}(D/p^{2n'}) \text{ for } n \text{ with } n > 2n_2 > 2n' \\ \qquad\qquad\qquad\qquad \text{if } n_2 > n' \text{ with } p^{2n'}|D, \\ p^n \text{ for } n \le 2n' \text{ with } p^{2n'}|D, \\ 0 \text{ otherwise} \end{cases}$$

where $\phi_{p,n_2-n'} = p^{n_2-n'}/2$ if $0 \le n' < n_2$, and 1 if $n' = n_2$, respectively.

Proof is elementar.

Examples:

A) $\sharp\{\lambda \mod 3^8 | (45\lambda+18)^2 \equiv 0 \mod 3^8\} = \sharp\{\lambda \mod 3^8 | \lambda^2 \equiv 0 \mod 3^4\}$
$= 3^4 \cdot \sharp\{\lambda \mod 3^4 | \lambda^2 \equiv 0 \mod 3^4\}$

B) a) $\sharp\{\lambda \mod 3^8 | (5^4\lambda+3^2)^2 \equiv D \mod 3^8\} = \sharp\{\lambda \mod 3^8 | \lambda^2 \equiv D \mod 3^8\}$,

b) $\sharp\{\lambda \mod 3 | (3\lambda+1)^2 \equiv D \mod 3\} = \frac{3}{2}\sharp\{\lambda \mod 3 | \lambda^2 \equiv D \mod 3\}$,

c) $\sharp\{\lambda \mod 3^6 | (3^6\lambda+9)^2 \equiv D \mod 3^6\} = \sharp\{\lambda \mod 3^6 | (3^4\lambda+1)^2 \equiv D/3^4 \mod 3^2\} = 3^4\frac{3}{2}\sharp\{\lambda \mod 3^2 | \lambda^2 \equiv D/3^4 \mod 3^2\}$ if $3^4|D$,

d) $\sharp\{\lambda \mod 3^4 | (3^6\lambda+9)^2 \equiv D \mod 3^4\} = 3^4$ if $3^4|D$,

e) $\sharp\{\lambda \mod 3^6 | 3^4(3^4\lambda+1)^2 \equiv D \mod 3^6\} = 0$ if $3 \nmid D$,

f) $\sharp\{\lambda \mod 3^4 | (3^6\lambda+9)^2 \equiv D \mod 3^4\} = 0$ if $3^4 \nmid D$.

Using this lemma and substituting $n - 2n_2 \to n$ we gain for the factor $Z^{D\le 0}_{m,t_1-r_1,p}(s)$ following sum:

(2,18)

$$\begin{aligned} Z^{D\le 0}_{m,t_1-r_1,p}(s) &= \sum_{0\le n\le 2n'-1} \frac{1}{p^{n(s-1)}} + p^{2n'(1-s)}\phi_{p,n_2-n'} \sum_{n\ge 0} \frac{N_{p^n}(\frac{D}{4mp^{2n'}})}{p^{ns}} \\ &= \frac{\zeta_p(s-1)}{\zeta_p(2n'(s-1))} + p^{2n'(1-s)}\phi_{p,n_2-n'} \frac{\zeta_p(s)}{\zeta_p(2s)} L_{\Delta_p}(s) \text{ if } p^{2n'}|D \end{aligned}$$

where $\Delta_p = \frac{D}{4mp^{2n'}}$ with $p|m_2$. Here we have used $N_{p^n}(D/4m) = p^n$ if $0 \le n \le 2n'$ and

$$\sum_{0\le n\le 2n'-1} \frac{1}{p^{n(s-1)}} = \frac{1-p^{2n'(1-s)}}{1-p^{1-s}} = \frac{\zeta_p(s-1)}{\zeta_p(2n'(s-1))}.$$

The last line of (2,18) is equal to $\frac{\zeta_p(s)}{\zeta_p(2s)} L_{D/4m}(s)$ if $n_2 = n' = 0$.

We summarize above results in

Proposition 2. *For $n_2 := ord_p m_2$, $n' := ord_p(t_1 - r_1)$ $(n_2 \geq n')$ and*

(2,19a)

$$\psi_{t,r}^{D\leq 0}(s) = \zeta(2s-1/2)^{-1} \sum_{b\geq 1} \frac{N_{b,m,t_1-r_1}(D/4m)}{b^{2s-1/2}} = \zeta(2s-1/2)^{-1} \prod_p Z_{m,t_1-r_1,p}^{D<0}(2s-1/2)$$

we have

$$Z_{m,t_1-r_1,p}^{D\leq 0}(2s-1/2) = \begin{cases} \dfrac{\zeta_p(2s-3/2)}{\zeta_p(2n_2(2s-3/2))} + \phi_{p,n_2} p^{2n_2(\frac{3}{2}-2s)} \dfrac{\zeta_p(2s-1/2)}{\zeta_p(4s-1)} L_{\Delta_p}(2s-1/2) & \text{if } p^{2n'} | D \\ 0 & \text{if } p^{2n'} \nmid D. \end{cases}$$

where $\Delta_p := \frac{D}{4mp^{2n'}}$ and $\phi_{p,n_2-n'} = \frac{p^{n_2-n'}}{2}$ if $0 \leq n' < n_2$, and 1 if $n' = n_2$, respectively. Here $L_{\Delta_p}(s) = \zeta_p(2s-1)$ if $D = 0$. Especially, if $n_2 = n' = 0$ we have

$$Z_{m,t_1-r_1,p}^{D\leq 0}(2s-1/2) = \frac{\zeta_p(2s-1/2)}{\zeta_p(4s-1)} L_{D/4m}(2s-1/2).$$

According to (2,14) and the prop.2 we obtain prop.1 in **1**.

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